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A Compatibility Equation for Nonequilibrium Ionization

F. H. Shair*
General Electric Company, Philadelphia, Pa.

An expression of compatibility, independent of the electron collision frequency, is developed from the electron heating equation (discussed in Ref. 1) and the Saha equation. By using this compatibility relation, one may compute the profiles of electron temperature, plasma electrical conductivity, and electric field, between the anode and cathode of moderate pressure plasma devices (such as a diode or an MHD generator), once the neutral particle temperature profile and the current density have been measured. The current passing through any plane parallel to the electrodes is assumed constant. The influence of nonuniformities due to insulator walls is neglected. Radiation losses are neglected. As a first approximation, electron diffusion is also neglected.

Starting with the electron heating equation in which very steep spatial gradients are neglected,

$$j^2/\sigma = n_e\nu_e(2m_e/m_n)\delta(\tfrac{3}{2}kT_e - \tfrac{3}{2}kT_n) \qquad (1)$$

where j is the current density, σ the plasma scalar electrical conductivity, n_e the electron density, ν_e the electron collision frequency, and m_e the electric mass. m_n is the average heavy particle mass ($2m_e/m_n$ represents the average fraction of energy loss per electron per elastic heavy particle collision), δ the loss factor that accounts for inelastic collisions and radiation losses (δ is near unity for monatomic particles but is very large for polyatomic particles, as discussed in Ref. 2), k the Boltzmann constant, T_e the average electron temperature, and T_n the average neutral particle temperature.

The scalar conductivity equation is

$$\sigma = (n_e e^2/m_e\nu_e) \qquad (2)$$

where e is the electron charge.

The electron collision frequency is eliminated by combining Eqs. (1) and (2). The resulting expression is then substituted into the Saha equation:

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* Research Engineer, Space Sciences Laboratory, Missile and Space Division.

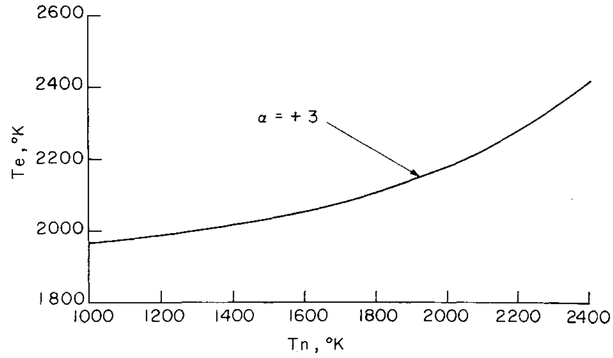

Fig. 1. Electron temperature vs neutral particle temperature for α = 3.

$$\frac{n_e^2}{n_s} = \left(\frac{2\pi m_e kT_e}{h^2}\right)^{3/2} e^{-E_0/kT_e} \qquad (3)$$

where n_s is the number density of ionizable particles (in the case of a seeded plasma, $n_s = X_s n_n$, where X_s is the mole fraction of seed in n_n neutral particles), h is the Planck constant, and E_0 is the ionization potential. Note that Eq. (3) is valid when $n_e \ll n_s$ and when the ratio of the statistical weights is equal to unity.

This resulting expression (the compatibility equation) is found to be

$$j^2 = \left(\frac{3_e{}^2\delta}{m_n}\right)(X_sP_n)\left(\frac{T_e}{T_n} - 1\right)\left(\frac{2\pi m_e kT_e}{h^2}\right)^{3/2} e^{-E_0/kT_e} \qquad (4)$$

For convenience of computation, Eq. (4) is rearranged:

$$\frac{5040\ E_0}{T_e} - \log\left(\frac{T_e}{T_n} - 1\right) - 1.5\log T_e = 8.05107 - \alpha \qquad (5)$$

where

α = $\log(j^2M_n/\delta X_sP_n)$
j = current density, amp/cm^2
M_n = average neutral particle atomic weight
P_n = static pressure, atm
T_n = neutral particle static temperature, °K
T_e = average electron temperature, °K
E_0 = ionization potential, ev

Consequently, when α and $T_n(x)$ are measured, $T_e(x)$ can be computed from Eq. (4). (For convenience, a plot of T_n

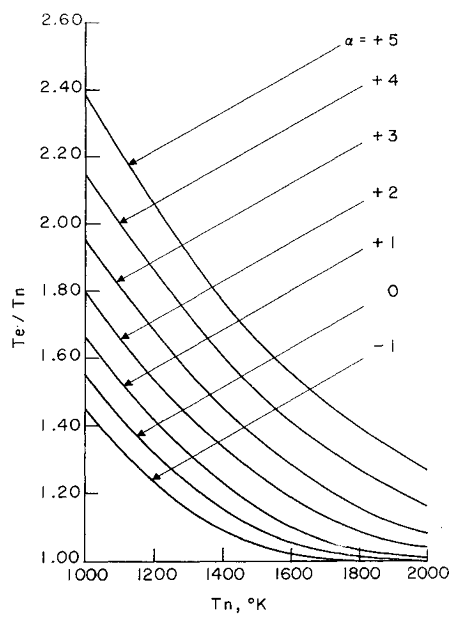

Fig. 2 Temperature ratio vs neutral particle temperature for various α.

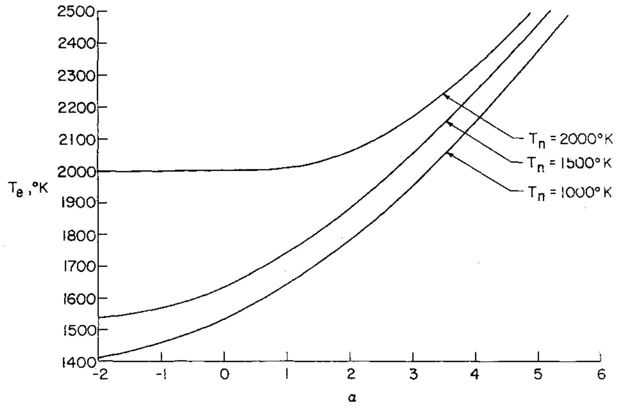

Fig. 3 T_e vs α for various neutral particle temperatures.

vs T_e can be computed for various T_e.) Here, x represents the normal distance from either of the electrodes. The profile of T_e can then be used to compute $n_e(x)$, $\sigma(x)$, and $E(x)$ between the electrodes. The more positive α, the more out of equilibrium are the electrons from the heavy particles. Figure 1 shows T_e vs T_n necessary for $\alpha = +3$ (if $M_n \sim 40$, $X_s \sim 4 \times 10^{-3}$ and $P_n \sim 1$ atm, the j would be around 0.3 amp/cm²). It is interesting to note that the electron temperature boundary layer is usually less steep than that of the heavy particles. However, this electron temperature profile can still create a variation in the scalar conductivity, which will in turn create apparent electrical sheaths near the electrodes and thus reduce the main stream electric field. Factors that reduce δ (such as impurities raising the value of δ) will make this effect more severe. Figure 2 is a typical convenient graphical representation of the compatibility equation which can be used to quickly compute $T_e(x)$ once $T_n(x)$ and α are measured. Figures 3 and 4 show the functional dependence of T_e and $n_e^2/(n_s - n_e)$ upon α, respectively.

A corresponding expression is derived for the electron density. Recall the logarithmic form of the Saha equation:

$$\frac{5040\ E_0}{T_e} - 1.5 \log T_e = 15.38274 - \log \frac{n_e^2}{n_s - n_e} \quad (6)$$

The logarithmic form of the compatibility equation is rearranged:

$$\frac{5040\ E_0}{T_e} - 1.5 \log T_e = 8.05107 - \alpha + \log\left(\frac{T_e}{T_n} - 1\right) \quad (5')$$

The ideal gas law is used:

$$n_s = 0.734 \times 10^{22}(X_s P_n/T_n) \quad (7)$$

Note that

$$\alpha \equiv \log_{10}(j^2 M_n/X_s P_n)$$

Then, for the condition $n_e \ll n_s$, Eqs. (5) and (6) are combined to yield

$$n_e = 4.0 \times 10^{14}\left(\frac{j^2 M_n}{\delta T_n}\right)^{1/2} 10^{-1/2 \log[(T_e/T_n) - 1]}, \ \frac{\#}{\text{cm}^3} \quad (8)$$

where j is in amperes per square centimeters and T_n is in degrees Kelvin. For the region $1.3 < T_e/T_n < 3.0$, Eq. (8) can be approximated within a factor of 2 by the following expression†:

$$n_e = 4.0 \times 10^{14} j (M_n/T_n)^{1/2} \qquad \#/\text{cm}^3 \quad (9)$$

† At $T_e/T_n = 2$, Eq. (8) reduces exactly to Eq. (9).

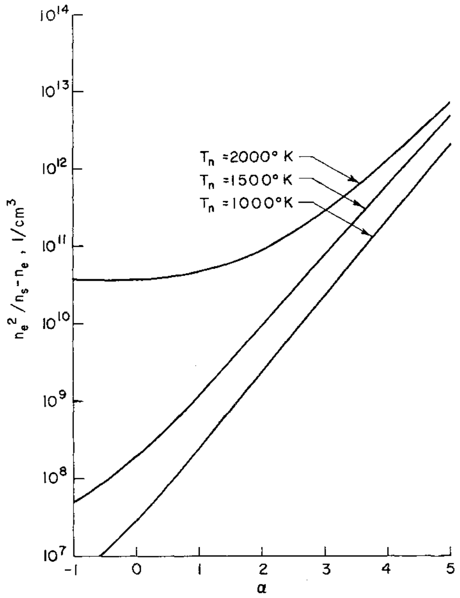

Fig. 4 $n_e^2/(n_s - n_e)$ vs α for various neutral particle temperatures.

The plasma scalar conductivity can then be calculated from the hard sphere model in which the electrons are assumed to have a Maxwellian distribution:

$$\sigma_p = \frac{4.528 \times 10^{-10} n_e}{T_e^{1/2} \sum_i n_i Q_{ei}} \ \frac{\text{mho}}{\text{cm}} \quad (10)$$

where Q_{ei} is the elastic electron collision cross section for the ith neutral particle species in square centimeters, T_e is in degrees Kelvin, n_e is in electrons per centimeters cubed, n_i is in electrons per centimeters cubed. Since $X_i \equiv n_i/n_n$ and $n_n = 0.734 \times 10^{22}(P_n/T_n)$, from the ideal gas law,

$$\sigma_p = \frac{4.528 \times 10^{-32}}{0.734}\left(\frac{T_n n_e}{T_e^{1/2} P_n \sum_i X_i Q_{ei}}\right) \quad (11)$$

Combining Eqs. (8) and (11),

$$\sigma_p = 2.47 \times 10^{-17} \frac{j}{P_n}\left(\frac{M_n T_n}{\delta T_e}\right)^{1/2} \frac{10^{-12 \log[(T_e/T_n) - 1]}}{\sum_i X_i Q_{ei}} \quad (12)$$

where σ_p is in ohms per centimeter.

Consequently, the associated electric field is

$$\frac{j}{\sigma_p} = 4.06 \times 10^{16} P_n \left(\frac{\delta T_e}{M_n T_n}\right)^{1/2} \frac{\sum_i X_i Q_{ei}}{10^{-1/2 \log[(T_e/T_n) - 1]}} \quad (13)$$

For $T_e/T_n = 2$, Eq. (13) reduces to

$$\frac{j}{\sigma_p} = 4.06 \times 10^{16} P_n \left(\frac{\delta T_e}{M_n T_n}\right)^{1/2} \sum_i X_i Q_{ei} \ \frac{\text{v}}{\text{cm}} \quad (14)$$

where P_n is in atmospheres, δ is dimensionless, T_e and T_n is in degrees Kelvin, X_i is dimensionless, and Q_{ei} is in square centimeters.

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